

Chapter 1

Critical Phenomena

The aim of this introductory chapter is to introduce the concept of a phase transition and to motivate the subject of statistical field theory. Here we introduce the concept of universality as applied to critical phenomena and define some of the notation used throughout these lectures.

1.1 Collective Phenomena: from Particles to Fields

It is rare in physics to find examples of *interacting* many-particle systems which admit to a full and accessible microscopic description. More useful is a *hydrodynamic* description of the **collective** long-wavelength behaviour which surrenders information at the microscopic scale. A familiar example is the Navier-Stokes equation of fluid dynamics. The averaged variables appropriate to these length and time scales are no longer the discrete set of particle degrees of freedom but rather slowly varying continuous fields describing the collective motion of a macroscopic set of particles. Familiar examples include magnetic spin-waves, and vibrational or phonon modes of an atomic lattice.

The most striking consequence of interactions among particles is the appearance of new phases of matter whose collective behaviour bears little resemblance to that of a few particles. How do the particles then transform from one macroscopic state to another? Formally, all macroscopic properties can be deduced from the free energy or the partition function. However, since phase transitions typically involve dramatic changes in response functions, they must correspond to singularities in the free energy. Since the canonical partition function of a finite collection of particles is always analytic, phase transitions can only be associated with infinitely many particles, i.e. the **thermodynamic limit**. The study of phase transitions is thus related to finding the origin of various singularities in the free energy and characterising them.

Consider the classical equilibrium statistical mechanics of a regular lattice of an **Ising ferromagnet** (i.e. spin degrees of freedom which can take only two values, ± 1 , with short range interactions that favour the spins to be aligned). When viewed microscopically, the development of magnetic moments on the atomic lattice sites of a crystal and the subsequent ordering of the moments is a complex process involving the cooperative phenomena

of many interacting electrons. However, remarkably, the thermodynamic properties of different macroscopic ferromagnetic systems are observed to be the same — e.g. temperature dependence of the specific heat, susceptibility, etc. Moreover, the thermodynamic critical properties of completely different physical systems, such as an Ising ferromagnet and a liquid at its boiling point, show the same dependence on, say, temperature. What is the physical origin of this **Universality**?

Suppose we take a ferromagnetic material and measure some of its material properties such as magnetisation. Dividing the sample into two roughly equal halves, keeping the internal variables like temperature and magnetic field the same, the macroscopic properties of each piece will then be the same as the whole. The same holds true if the process is repeated. But eventually, after many iterations, something different must happen because we know that the magnet is made up of electrons and ions. The characteristic length scale at which the overall properties of the pieces begins to differ markedly from those of the original system defines a **correlation length**. It is the typical length scale over which the fluctuations of the microscopic degrees of freedom are correlated.

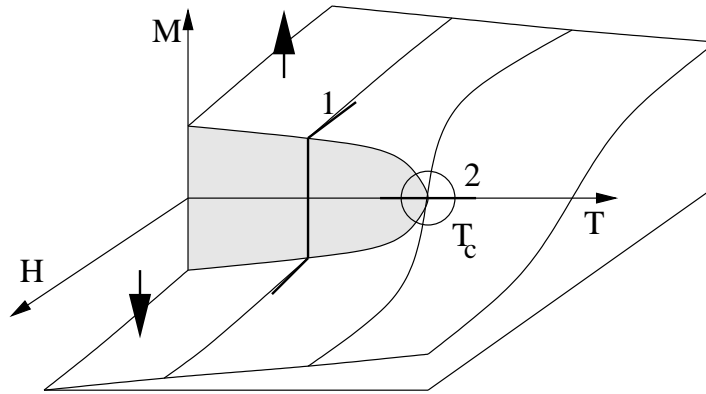


Figure 1.1: Phase diagram of the Ising ferromagnet showing the average magnetisation M as a function of magnetic field H and temperature T . Following trajectory 1 by changing the magnetic field at constant temperature $T < T_c$, the sample undergoes a first order phase transition from a phase in which the average magnetisation is positive (i.e. ‘spin-up’) to a phase in which the average is negative (i.e. ‘spin-down’). Secondly, by changing the temperature at fixed zero magnetic field, the system undergoes a second order phase transition at $T = T_c$ whereupon the average magnetisation grows continuously from zero. This second order transition is accompanied by a **spontaneous symmetry breaking** in which the system chooses to be in either an up or down-spin phase. (Contrast this phase diagram with that of the liquid-gas transition — magnetisation $M \rightarrow$ density ρ , and magnetic field $H \rightarrow$ pressure.) The circle marks the region in the vicinity of the critical point where the correlation length is large as compared to the microscopic scales of the system, and ‘Ginzburg-Landau theory’ applies.

Now experience tells us that a ferromagnet may abruptly change its macroscopic behaviour when the external conditions such as the temperature or magnetic field are varied. The points at which this happens are called **critical points**, and they mark a

phase transition from one state to another. In the ferromagnet, there are essentially two ways in which the transition can occur (see Fig. 1.1). In the first case, the two states on either side of the critical point (spin up) and (spin down) coexist at the critical point. Such transitions, involve **discontinuous** behaviour of thermodynamic properties and are termed **first-order** (cf. melting of a three-dimensional solid). The correlation length at such a first-order transition is generally finite.

In the second case, the transition is **continuous**, and the correlation length becomes effectively infinite. Fluctuations become correlated over all distances, which forces the whole system to be in a unique, critical, phase. The two phases on either side of the transition (paramagnetic and ferromagnetic) must become identical as the critical point is approached. Therefore, as the correlation length diverges, the magnetisation goes smoothly to zero. The transition is said to be **second-order**.

The divergence of the correlation length in the vicinity of a second order phase transition suggests that properties near the critical point can be accurately described within an effective theory involving only long-range collective fluctuations of the system. This invites the construction of a phenomenological Hamiltonian or Free energy which is constrained only by the fundamental symmetries of the system. Such a description goes under the name of **Ginzburg-Landau theory**. Although the detailed manner in which the material properties and microscopic couplings of the ferromagnet influence the parameters of the effective theory might be unknown, qualitative properties such as the **scaling** behaviour are completely defined.

From this observation, we can draw important conclusions: critical properties in the vicinity of classical and quantum second order phase transitions fall into a limited number of **universality classes** defined not by detailed material parameters, but by the fundamental symmetries of the system. When we study the critical properties of the transition in an Ising ferromagnet, we learn about the nature of the liquid-gas transition! (See below.) Similarly, in the jargon of statistical field theory, a superconductor, with its complex order parameter, is in the same universality class as the two-component or ‘XY Heisenberg’ ferromagnet. The analysis of critical properties associated with different universality classes is the subject of **Statistical field theory**.

1.2 Phase Transitions

With these introductory remarks in mind, let us consider more carefully the classic example of a phase transition involving the condensation of a gas into a liquid. The phase diagram represented in Fig. 1.2a exhibits several important and generic features of a second order phase transition:

1. In the (T, P) plane, the phase transition occurs along a line that terminates at a **critical point** (T_c, P_c) .
2. In the $(P, v \equiv V/N)$ plane, the transition appears as a **coexistence interval**, corresponding to a mixture of gas and liquid of densities $\rho_g = 1/v_g$ and $\rho_l = 1/v_l$ at temperatures $T < T_c$.

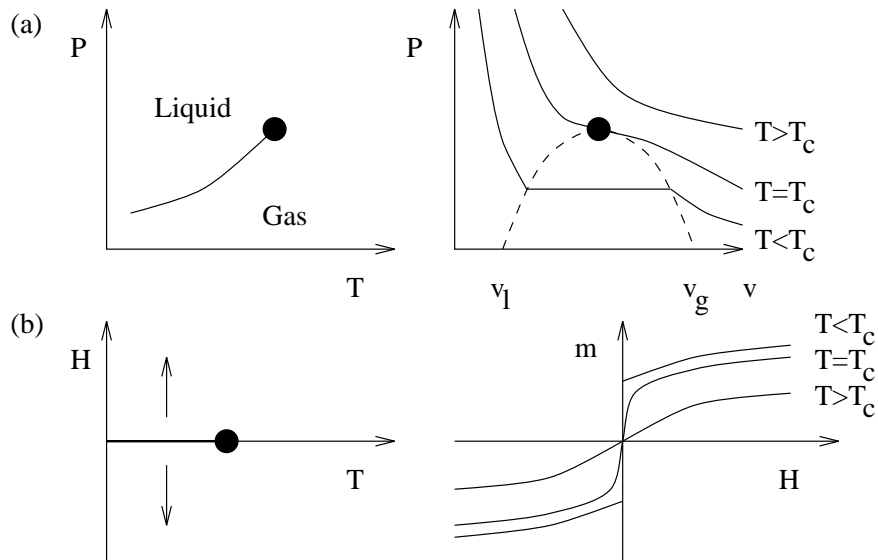


Figure 1.2: Phase diagrams of (a) the liquid-gas transition, and (b) the ferromagnetic transition. In each case the phase diagrams are drawn in two different planes. Notice the similarity between the two ($1/v \leftrightarrow m$, $P \leftrightarrow H$). Isotherms above, below, and at T_c are sketched.

3. Due to the termination of the coexistence line, it is possible to go from the gas phase to the liquid phase continuously (without a phase transition) by going around the critical point. Thus there are no fundamental differences between the liquid and gas phases (i.e. there is no change of fundamental symmetry).

From a mathematical perspective, the free energy of this system is an analytic function in the (P, T) plane except for some form of branch cut along the phase boundary. Observations in the vicinity of the critical point further indicate that:

4. The difference between the densities of the coexisting liquid and gas phases vanishes on approaching T_c , i.e. $\rho_l \rightarrow \rho_g$ as $T \rightarrow T_c^-$.
5. The pressure versus volume isotherms become progressively more flat on approaching T_c from the high temperature side. This implies that the **isothermal compressibility**, the rate of change of density with pressure, $\kappa_T = -(1/V)\partial V/\partial P|_T$ diverges as $T \rightarrow T_c^+$.
6. The fluid appears “milky” close to criticality. This phenomenon, known as **critical opalescence**, suggests the existence of collective fluctuations in the gas at long enough wavelengths to scatter visible light. These fluctuations must necessarily involve many particles, and a coarse-graining procedure must be appropriate to their description.

How does this phase diagram compare with the phase transition that occurs between paramagnetic and ferromagnetic phases of certain substances such as iron or nickel? These materials become spontaneously magnetised below a Curie temperature, T_c . Redrawn in cross-section, the phase diagram of Fig. 1.1 is shown in Fig. 1.2b. There is a discontinuity in magnetisation as the magnetic field, H goes through zero, and the magnetisation isotherms, $M(H)$ have much in common with the condensation problem. In both cases, a line of discontinuous transitions terminates at a critical point, and the isotherms exhibit singular behaviour in the vicinity of this point. The phase diagram is simpler in appearance because the symmetry $H \rightarrow -H$ ensures that the critical point occurs at $H_c = M_c = 0$.

In spite of the apparent similarities between the magnetic and liquid-gas transition, our intuition would suggest that they are quite different. Magnetic transitions are usually observed to be **second-order** — that is, the magnetisation m , which plays the role of an **order parameter**, rises continuously from zero below the transition. On the other hand, our everyday experience of boiling a kettle of water shows the liquid-gas transition to be **first-order** — that is, the order parameter, corresponding to the difference between the actual density and the density at the critical point, $\rho - \rho_c$, jumps discontinuously at the critical point with an accompanying absorption of latent heat of vapourisation (implying a discontinuous jump in the entropy of the system: $Q_L = T_c \Delta S$).

However, the perceived difference in behaviour simply reflects different paths through the transition in the two cases. In a ferromagnet, the natural experimental path ($b \rightarrow c \rightarrow d$ in Fig. 1.3a) is one in which the external magnetic field takes the value $H = 0$. For $T > T_c$, the average magnetisation is zero, while for $T < T_c$ the magnetisation grows continuously from zero. In a liquid, the natural path is one in which temperature is varied at constant pressure ($b' \rightarrow c' \rightarrow d'$ in Fig. 1.3b). Along this path, there is a discontinuous change in the density. This is the first-order boiling transition.

A path in the ferromagnetic (H, T) plane analogous to the constant pressure path in a fluid is shown in Fig. 1.3c. Along this path m is negative from $b' \rightarrow c'$ and then jumps discontinuously to a positive value as the coexistence line is crossed and remains positive from $c' \rightarrow d'$. It is clear that the path in a fluid that most closely resembles the $H = 0$ path in a magnet, which shows a second-order transition, is the one with density fixed at its critical value ρ_c , i.e. the **critical isochore** ($b \rightarrow c \rightarrow d$ in Fig. 1.3d).

†**The Lattice-Gas model:** We can directly show the equivalence between the Ising ferromagnet and a lattice gas model. Consider a classical three dimensional gas of N particles in a volume V and subject to the Hamiltonian

$$H = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{ij} u(\mathbf{r}_i - \mathbf{r}_j).$$

By integrating over the phase space configurations of the particles, the grand partition function $\Theta = \text{tr } e^{-\beta(H - \mu N)}$ can be written as

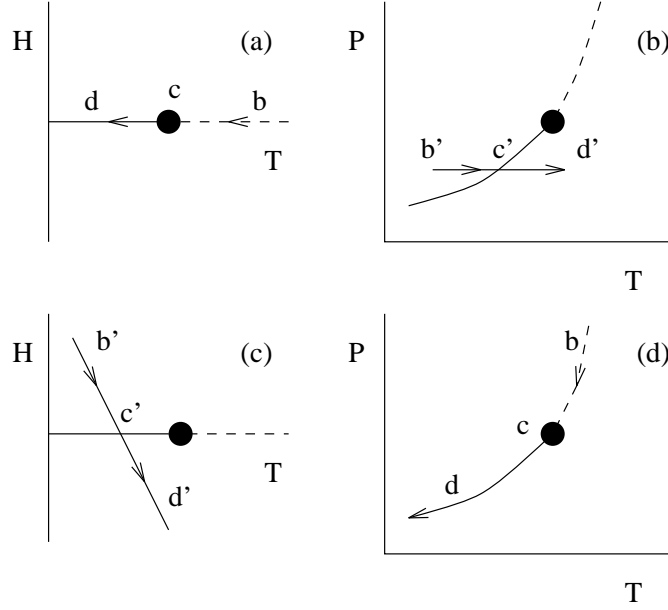


Figure 1.3: Comparison of the ferromagnetic phase transition with the liquid-gas transition. The different paths identified in the figure are discussed in the text.

$$\Theta = \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{e^{\beta\mu}}{\lambda^3} \right)^N \int \prod_{i=1}^N d^3\mathbf{r}_i \exp \left[-\frac{\beta}{2} \sum_{ij} u(\mathbf{r}_i - \mathbf{r}_j) \right],$$

where μ denotes the chemical potential, and $\lambda = h/(2\pi mkT)^{1/2}$.

The volume V is now subdivided into $N_s = V/a^3$ cells of volume a^3 (N_s is not to be confused with N), with spacing a chosen small enough so that each cell α is either empty or occupied by *one* particle; i.e., the cell occupation number n_α is restricted to 0 or 1 ($\alpha = 1, 2, \dots, N_s$). After approximating the integrals $\int d^3\mathbf{r}$ by sums $a^3 \sum_{\alpha=1}^{N_s}$, the grand canonical partition function becomes

$$\Theta \approx \sum_{\{n_\alpha=0,1\}} \left(\frac{a^3 e^{\beta\mu}}{\lambda^3} \right)^{\sum_\alpha n_\alpha} \exp \left[-\frac{\beta}{2} \sum_{\alpha,\beta} n_\alpha n_\beta u(\mathbf{r}_\alpha - \mathbf{r}_\beta) \right].$$

By setting $n_\alpha = (1 + \sigma_\alpha)/2$ and approximating the potential by $u(\mathbf{r}_\alpha - \mathbf{r}_\beta) = -J$ for nearest-neighbours only, the model becomes identical to the Ising ferromagnet with discrete spins $\sigma_\alpha \pm 1$.

$$\Theta = \sum_{\sigma_\alpha \pm} \exp \left[h \sum_\alpha \sigma_\alpha - \frac{\beta J}{4} \sum_{\langle \alpha\beta \rangle} \sigma_\alpha \sigma_\beta \right], \quad (1.1)$$

where $h = \ln\left(\frac{a^3}{\lambda^3}\right) + \beta\mu - \frac{3}{2}J\beta$ is the effective magnetic field. Using Maxwell's relation

$$\left(\frac{\partial\mu}{\partial V}\right)_{N,T} = -\left(\frac{\partial p}{\partial N}\right)_{V,T}, \quad (1.2)$$

and from the fact that in the thermodynamic limit the variables are only functions of the particle density $\rho = \frac{N}{V}$, we can relate the isothermal compressibility to the ferromagnetic susceptibility

$$\rho\left(\frac{\partial\rho}{\partial p}\right)_T = \left(\frac{\partial\rho}{\partial\mu}\right)_T = \frac{2V}{T}\left(\frac{\partial m}{\partial h}\right)_T \sim (T - T_c)^{-\gamma}, \quad (1.3)$$

which has a power law divergence at the critical temperature $T = T_c$. The magnetisation m is given by $m = 2\rho V - N_s$. Note that the left and right hand sides of the above equation are evaluated at $h = 0$. In fact, since the chemical potential $\mu(T, p)$ can be expressed as a function of temperature and pressure, the following relation serves to define p_c at each temperature T

$$h = \ln\left(\frac{a^3}{\lambda^3}\right) + \beta\mu(T, p_c) - \frac{3}{2}J\beta = 0. \quad (1.4)$$

Using the relation $\left(\frac{\partial\mu}{\partial p}\right)_T = \frac{1}{\rho}$, we can expand h around $p = p_c$

$$h = \frac{\beta}{\rho_c}(p - p_c). \quad (1.5)$$

Further, note that the spontaneous magnetisation can be related to the shape of the *coexistence curve*

$$\rho_g - \rho_l = \frac{m}{V} \sim (T_c - T)^\beta, \quad (1.6)$$

which rises with a power law below the critical temperature.

Comment: This type of manifest equivalence is usually quite straightforward. The equivalence premised on the Universality Principle, with respect to quantitative critical properties, connects systems which cannot be straightforwardly mapped onto each other.

1.3 Critical Behaviour

The singular behaviour in the vicinity of the critical point is characterised by a set of **critical exponents**. These exponents describe the non-analyticity of various thermodynamic functions. Remarkably transitions as different as the liquid/gas and ferromagnetic transition can be described by the same set of critical exponents and are said to belong to the same **Universality class**.

1.3.1 Significance of Power laws

▷ INFO: Prior to defining physically relevant critical exponents, we take a mathematical digression in probability theory. First consider an exponential probability density function

$$p(x) = ae^{-ax}$$

for $x > 0$. The mean $\langle x \rangle = 1/a$ and mean squared $\langle x^2 \rangle = 2/a$. So, for this distribution the *typical* scale is $1/a$ with typical fluctuations of $1/a$. Now, consider a power law distribution

$$p(x) \sim \frac{1}{x^{1+\mu}}.$$

For $\mu \leq 1$ the mean diverges $\langle x \rangle \rightarrow \infty$, which implies that there is no typical size. Also for $\mu \leq 2$, $\langle x^2 \rangle \rightarrow \infty$ so fluctuations are also unbounded. Such power laws distributions have no typical size and are thus scale invariant. They also have fluctuations or structure at all length scales.

Those critical exponents most commonly encountered are defined below.

1.3.2 Order Parameter

By definition, there is more than one equilibrium phase on a coexistence line. As mentioned above, the **order parameter** is a thermodynamic function that is different in each phase, and hence can be used to distinguish between them. For a (uniaxial) magnet, the order parameter is provided by the total magnetisation $M(H, T)$, or magnetisation density,

$$m(H, T) = \frac{M(H, T)}{V}.$$

In zero field, m vanishes for a paramagnet and is non-zero in a ferromagnet (see Fig. 1.4), i.e.

$$m(T, H \rightarrow 0^+) \propto \begin{cases} 0 & T > T_c, \\ |t|^\beta & T < T_c, \end{cases}$$

where $t = (T - T_c)/T_c$ denotes the **reduced temperature**. The singular behaviour of the order parameter along the coexistence line is therefore characterised by a critical exponent β . The singular behaviour of m along the critical isotherm is governed by another exponent, δ (see Fig. 1.2)

$$m(T = T_c, H) \propto H^{1/\delta}.$$

The two phases along the liquid-gas coexistence line are distinguished by their density allowing one to define $\rho - \rho_c$, where ρ_c denotes the critical density, as the order parameter.

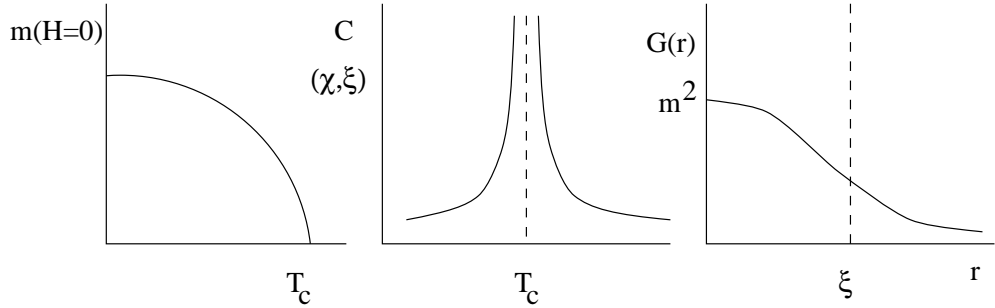


Figure 1.4: Critical behaviour of the magnetisation and response functions close to the ferromagnetic transition.

1.3.3 Response Functions

The critical system is highly sensitive to external perturbations: for example, at the liquid-gas critical point, the compressibility $\kappa_T = -(1/V)\partial V/\partial P|_T$ becomes infinite. The divergence of the compressibility or, more generally, the **susceptibility** of the system (i.e. the response of the order parameter to a field conjugate to it) is characterised by another critical exponent γ . For example, in a magnet, the analogous quantity is the zero-field **susceptibility**

$$\chi_{\pm}(T, H \rightarrow 0^+) = \left. \frac{\partial m}{\partial H} \right|_{H=0^+} \propto |t|^{-\gamma_{\pm}},$$

where, in principle, two exponents γ_+ and γ_- are necessary to describe the divergences on the two sides of the phase transition. Actually, in almost all cases, the same singularity governs both sides, and $\gamma_+ = \gamma_- = \gamma$.

Similarly, the **heat capacity** represents the thermal response function, and its singularities at zero field are described by the exponent α ,

$$C_{\pm} = \frac{\partial E}{\partial T} \propto |t|^{-\alpha_{\pm}},$$

where E denotes the internal energy and, again, the exponents usually coincide $\alpha_+ = \alpha_- = \alpha$.

1.3.4 Long-range Correlations

Since the response functions are related to equilibrium fluctuations, their divergence in fact implies that fluctuations are correlated over long distances. To see this let us consider the magnetic susceptibility of, say, the **Ising ferromagnet**. The latter describes a lattice of scalar or two-valued spins which interact ferromagnetically with their neighbours. Starting with the microscopic Ising Hamiltonian

$$H_{\text{Ising}} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j,$$

where $\{\sigma_i = \pm 1\}$ denotes the set of Ising spins, $M = \sum_i \sigma_i$ represents the total magnetisation, and the sum $\sum_{\langle ij \rangle}$ runs over neighbouring lattice sites. The total partition function takes the form¹

$$\mathcal{Z}(T, h) = \sum_{\{\sigma_i\}} e^{-\beta(H_{\text{Ising}} - hM)}.$$

Here we have included an external magnetic field h , and the sum extends over the complete set of microstates $\{\sigma_i\}$. From \mathcal{Z} , the thermal average magnetisation can be obtained from the equation

$$\langle M \rangle \equiv \frac{\partial \ln \mathcal{Z}}{\partial(\beta h)} = \frac{1}{\mathcal{Z}} \sum_{\{\sigma_i\}} M e^{-\beta(H_{\text{Ising}} - hM)}.$$

Taking another derivative one obtains the susceptibility

$$\chi(T, h) = \frac{1}{V} \frac{\partial \langle M \rangle}{\partial h} = \frac{\beta}{V} \left\{ \frac{1}{\mathcal{Z}} \sum_{\{\sigma_i\}} M^2 e^{-\beta(H_{\text{Ising}} - hM)} - \left(\frac{1}{\mathcal{Z}} \sum_{\{\sigma_i\}} M e^{-\beta(H_{\text{Ising}} - hM)} \right)^2 \right\}$$

which gives the relation

$$\frac{V\chi}{\beta} = \text{var}(M) \equiv \langle M^2 \rangle - \langle M \rangle^2.$$

Now the overall magnetisation can be thought of as arising from separate contributions from different parts of the system, i.e. taking a continuum limit

$$M = \int d\mathbf{x} m(\mathbf{x}),$$

where $m(\mathbf{x})$ represents the ‘local’ magnetisation. Substituting this definition into the equation above we obtain

$$k_B T \chi = \frac{1}{V} \int d\mathbf{x} \int d\mathbf{x}' [\langle m(\mathbf{x})m(\mathbf{x}') \rangle - \langle m(\mathbf{x}) \rangle \langle m(\mathbf{x}') \rangle].$$

Since the system is symmetric under spatial translation, $\langle m(\mathbf{x}) \rangle$ is a constant m independent of position, while $\langle m(\mathbf{x})m(\mathbf{x}') \rangle = G(\mathbf{x} - \mathbf{x}')$ depends only on the spatial separation. Thus, in terms of the ‘**connected**’ correlation function defined as $G_c(\mathbf{x}) \equiv \langle m(\mathbf{x})m(0) \rangle_c \equiv \langle m(\mathbf{x})m(0) \rangle - m^2$, the susceptibility is given by

$$k_B T \chi = \int d\mathbf{x} \langle m(\mathbf{x})m(0) \rangle_c.$$

¹Throughout these notes $1/k_B T$ and the symbol β (not to be mistaken for the order parameter exponent) will be used interchangeably.

The connected correlation function is a measure of how the local fluctuations in one part of the system affect those in another. Typically such influences occur over a characteristic distance ξ known as the **correlation length** (see Fig. 1.4). (In many cases, the connected correlation function $G_c(\mathbf{x})$ decays exponentially $\exp[-|\mathbf{x}|/\xi]$ at separations $|\mathbf{x}| > \xi$.) If $g \sim m^2$ denotes a typical value of the correlation function for $|\mathbf{x}| < \xi$, it follows that $k_B T \chi < g \xi^d$ where d denotes the dimensionality of the system; the divergence of χ necessarily implies the divergence of ξ . This divergence of the correlation length also explains the observation of **critical opalescence**. The correlation function can be measured by scattering probes, and its divergence

$$\xi_{\pm}(T, H = 0) \propto |t|^{-\nu_{\pm}}$$

is controlled by exponents $\nu_+ = \nu_- = \nu$.

This completes our preliminary survey of phase transitions and critical phenomena. We found that the singular behaviour of thermodynamic functions at a critical point (the termination of a coexistence line) can be characterised by a set of critical exponents. Experimental observations indicate that these exponents are **universal**, independent of the material, and to some extent of the nature of the transition. Moreover the divergence of the response functions, as well as fluctuations, indicate that correlations become long-ranged in the vicinity of this point. The fluctuations responsible for the correlations involve many particles, and their description calls for a “coarse-graining” approach. In the next chapter we will exploit this idea to construct a **statistical field theory** which reveals the origin of the universality. To do so it will be convenient to frame our discussion in the language of magnetic systems whose symmetry properties are transparent. The results, however, have a much wider range of validity.