Derivation of Ginzburg-Landau Hamiltonian for the Ising model

$$H_{Ising} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

Partition function $Z = \mathrm{Tr}\, e^{-\beta H_{Ising}}$

$$Z = \mathrm{Tr}_s\, e^{\frac{1}{2} \sum_{r,r'} K(r,r') s(r) s(r') + \sum h(r) s(r)}$$

Hubbard-Stratonovich transformation

This generalized the simple Gaussian integral

$$\int_{-\infty}^{\infty} d\varphi\, e^{-\frac{1}{2}\varphi K^{-1} \varphi + \varphi s}$$

$$= \int d\varphi\, e^{-\frac{1}{2}(\varphi - sK) K^{-1} (\varphi - Ks) + \frac{1}{2} sKs}$$

$$= \int d\varphi'\, e^{-\frac{1}{2} \varphi' K^{-1} \varphi' + \frac{1}{2} sKs}$$

$$= \sqrt{2\pi K}\, e^{\frac{1}{2} sKs}$$

to the multi-variable case when K is a matrix, labelled by r, r'

$$\int \prod_r d\varphi(r)\, e^{-\frac{1}{2}\sum_{r,r'} \varphi(r) K^{-1}(r,r')\varphi(r') + \sum_r \varphi(r) s(r)}$$

$$\propto \sqrt{\det K}\;\; e^{\frac{1}{2}\sum_{r,r'} s(r) K(r,r') s(r')}$$

$$Z = \operatorname{Tr}_s \int \prod_r d\varphi(r)\, e^{-\frac{1}{2}\sum_{r,r'} \varphi(r) K^{-1}(r,r')\varphi(r') + \sum_r (\varphi(r)+h(r)) s(r)}$$

$$\sum_{s=\pm 1} e^{(\varphi+h)s} \propto \cosh(\varphi(r)+h(r))$$

$$Z \propto \int \prod_r d\varphi(r)\, e^{-\frac{1}{2}\sum_{r,r'} \varphi(r) K^{-1}(r,r')\varphi(r') + \sum_r \log\cosh(\varphi+h)}$$

— Euclidean field theory

$K(r,r') = K(r-r')$ — lattice translation symmetry

$$K(r-r') = \int \frac{d^d k}{(2\pi)^d}\, e^{ik(r-r')}\, \tilde{K}(k)$$

$$K^{-1}(r-r') = \int_{1BZ} \frac{d^d k}{(2\pi)^d} e^{ik(r-r')} \frac{1}{\hat{K}(k)}$$

$$\frac{1}{2}\sum_{r,r'} \varphi(r) K^{-1}(r-r')\varphi(r') = \frac{1}{2}\int_{BZ} \frac{d^d k}{(2\pi)^d} \hat{\varphi}(k)^* \frac{1}{\hat{K}(k)} \hat{\varphi}(k)$$

Suppose $K(r-r')$ is short-ranged

$$\hat{K}(k) = \hat{K}(0)\left[1 - \underset{\substack{\uparrow \\ \text{range of interaction}}}{R^2} k^2 + O(k^4)\right]$$

$$\hat{K}^{-1} = \frac{1}{\hat{K}(0)}\left[1 + R^2 k^2 + O(k^4)\right]$$

$$\sum_r \longleftrightarrow \int \frac{d^d r}{\underbrace{a^d}_{\text{lattice spacing}}}$$

$$k^2 \longleftrightarrow -\nabla^2$$

$$Z \propto \int \mathcal{D}\varphi\, e^{-\int \frac{d^d r}{a^d}\left[\frac{1}{2}\frac{1}{\hat{K}(0)}\varphi(1-R^2\nabla^2)\varphi\right] + \int \frac{d^d r}{a^d} \log\cosh(\varphi+h)}$$

Rescale $\varphi^2 \to \frac{\hat{K}(0)\,\varphi^2 a^d}{R^2}$

$$Z \propto \int \mathcal{D}\varphi \, e^{-\int d^d r \left[\frac{1}{2}(\underline{\nabla}\varphi)^2 + V(\varphi) \right]}$$

$$V(\varphi) = \frac{1}{2R^2}\varphi^2 - \frac{1}{a^d} \log \cosh\left[\sqrt{\frac{\hat{K}(0)\, a^d}{R^2}}\, \varphi + h \right]$$

$$\simeq \frac{1}{2R^2}\left(1 - \hat{K}(0)\right)\varphi^2 + \mathcal{O}\left(\frac{\hat{K}(0)^2 a^d}{R^4}\right)\varphi^4 + \cdots$$