† Mathematical digression

† means non-examinable.

Power laws ~ scale invariance

$X \sim \text{Bin}$

$$P_T(x) = \frac{1}{2^T} \binom{T}{\frac{x+T}{2}}$$

Taking continuum limit + CLT

$$P_t(x)_{dx} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

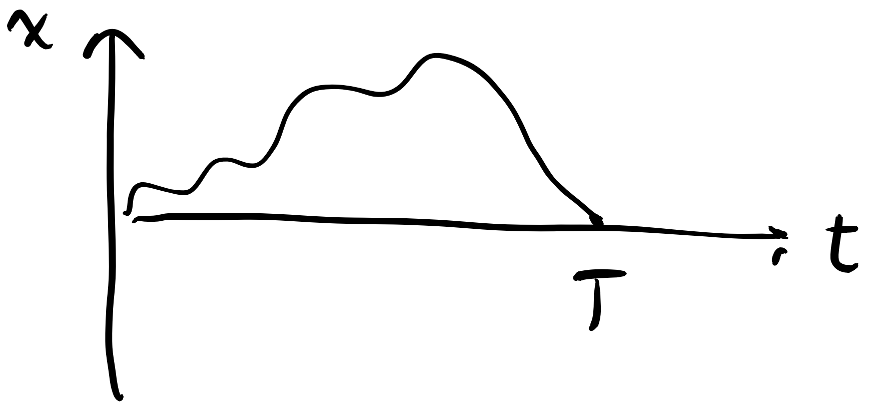
Rescale

$x' = bx$

$$P_t(x')dx' = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x'^2}{2b^2 t}} \frac{dx'}{b}$$

$\tau = b^2 t \quad P_\tau(x')dx' = \dfrac{1}{\sqrt{2\pi\tau}} e^{-\frac{x'^2}{2\tau}} dx'$

scale invariant



$$p = \frac{2}{2^{2T}}\left[\binom{2T-2}{T-1} - \binom{2T-2}{T-2}\right]$$

$$\to \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{(2T)^{\frac{3}{2}}} \quad \text{as } T \to \infty$$

$$\langle T \rangle \sim \int^{\infty} dT \, \frac{T}{T^{\frac{3}{2}}} \to \infty \, !$$

— gambler's ruin.

Consider, first, an exponential distribution

$$p(x) = a e^{-ax} \quad ; \ x \geq 0$$

Typical $x \sim \frac{1}{a}$

$$\langle x \rangle = \frac{1}{a} \quad ; \ \langle x^2 \rangle = \frac{2}{a^2}$$

Now power law

$$p(x) \sim \frac{1}{x^{1+\mu}}$$

$$\langle x \rangle \to \infty \text{ if } \mu \leq 1$$

i.e. no mean — No typical scale

$$\langle x^2 \rangle \to \infty \text{ if } \mu \leq 2$$

i.e. fluctuations are unbounded.

No variance.

Power law distributions imply structure or fluctuations on all length scales.