4.5 Answers

1. Making use of the rules of Grassmann algebra,

(a) $$\langle\eta|a_i^\dagger = \langle 0|\exp[-\sum_j a_j\bar\eta_j]a_i^\dagger = \langle 0|\prod_j(1-a_j\bar\eta_j)a_i^\dagger = \langle 0|(1-a_i\bar\eta_i)a_i^\dagger\prod_{j\neq i}(1-a_j\bar\eta_j)$$
$$= \underbrace{\langle 0|a_ia_i^\dagger}_{=\langle 0|[a_i,a_i^\dagger]_+=\langle 0|}\bar\eta_i\prod_{j\neq i}(1-a_j\bar\eta_j) = \langle 0|\prod_j(1-a_j\bar\eta_j)\bar\eta_i = \langle\eta|\bar\eta_i\,,$$

(b) $$a_i^\dagger|\eta\rangle = \underbrace{a_i^\dagger(1-\eta_ia_i^\dagger)}_{=a_i^\dagger=\partial_{\eta_i}\eta_ia_i^\dagger=-\partial_{\eta_i}(1-\eta_ia_i^\dagger)}\prod_{j\neq i}(1-\eta_ja_j^\dagger)|0\rangle = -\partial_{\eta_i}\prod_j(1-\eta_ja_j^\dagger)|0\rangle = -\partial_{\eta_i}|\eta\rangle\,,$$
$$\langle\eta|a_i = \langle 0|\prod_{j\neq i}(1-a_j\bar\eta_j)\underbrace{(1-a_i\bar\eta_i)a_i}_{=a_i=-\partial_{\bar\eta_i}a_i\bar\eta_i=\partial_{\bar\eta_i}(1-a_i\bar\eta_i)} = \partial_{\bar\eta_i}\langle 0|\prod_j(1-a_j\bar\eta_j) = \partial_{\bar\eta_i}\langle\eta|\,,$$

(c) $$\langle\eta|\nu\rangle = \langle\eta|\prod_i\underbrace{(1-\nu_ia_i^\dagger)}_{(1+a_i^\dagger\nu_i)}|0\rangle = \langle\eta|\prod_i\underbrace{(1+\bar\eta_i\nu_i)}_{\exp[\sum_i\bar\eta_i\nu_i]}|0\rangle = \exp[\sum_i\bar\eta_i\nu_i]\,.$$

(d) To prove the completeness of fermion coherent states, we apply Schur's lemma, i.e. we need to show that $[a_j^{(\dagger)}, \int d(\bar\eta,\eta)\; e^{-\sum_i\bar\eta_i\eta_i}|\eta\rangle\langle\eta|] = 0$.

$$a_j^\dagger\int d(\bar\eta,\eta)\; e^{-\sum_i\bar\eta_i\eta_i}|\eta\rangle\langle\eta| = -\int d(\bar\eta,\eta)\; e^{-\sum_i\bar\eta_i\eta_i}\partial_{\eta_j}|\eta\rangle\langle\eta|$$
$$= \int d(\bar\eta,\eta)\;\underbrace{\partial_{\eta_j}\left(e^{-\sum_i\bar\eta_i\eta_i}\right)}_{=\bar\eta_je^{-\sum_i\bar\eta_i\eta_i}}|\eta\rangle\langle\eta| = \int d(\bar\eta,\eta)\; e^{-\sum_i\bar\eta_i\eta_i}|\eta\rangle\langle\eta|a_j^\dagger$$

$$a_j\int d(\bar\eta,\eta)\; e^{-\sum_i\bar\eta_i\eta_i}|\eta\rangle\langle\eta| = \int d(\bar\eta,\eta)\quad\underbrace{e^{-\sum_i\bar\eta_i\eta_i}\eta_j}_{=-\partial_{\bar\eta_j}\left(e^{-\sum_i\bar\eta_i\eta_i}\right)}\quad|\eta\rangle\langle\eta|$$
$$= \int d(\bar\eta,\eta)\; e^{-\sum_i\bar\eta_i\eta_i}|\eta\rangle\partial_{\bar\eta_j}\langle\eta| = \int d(\bar\eta,\eta)\; e^{-\sum_i\bar\eta_i\eta_i}|\eta\rangle\langle\eta|a_j\,.$$

The constant of proportionality is fixed by taking the expectation value with the vacuum.

$$\langle 0|\int d(\bar\eta,\eta)\; e^{-\sum_i\bar\eta_i\eta_i}|\eta\rangle\langle\eta|0\rangle \;=\; \int d(\bar\eta,\eta)\; e^{-\sum_i\bar\eta_i\eta_i} = 1\,.$$

(e) A general n-particle state is given as $|n\rangle = a_1^\dagger\ldots a_n^\dagger|0\rangle$, $\langle n| = \langle 0|a_n\ldots a_1$, where we neglected a normalization factor. The matrix element $\langle n|\psi\rangle$, thus, reads

$$\langle n|\psi\rangle = \langle 0|a_n\ldots a_1|\psi\rangle = \langle 0|\psi_n\ldots\psi_1|\psi\rangle = \psi_n\ldots\psi_1\,.$$

Similarly, we obtain $\langle\psi|n\rangle = \bar\psi_1\ldots\bar\psi_n$. Using these results,

$$\begin{aligned}\langle n|\psi\rangle\langle\psi|n\rangle &= \psi_n\ldots\psi_1\bar\psi_1\ldots\bar\psi_n = \psi_1\bar\psi_1\ldots\psi_n\bar\psi_n\\ &= (\zeta\bar\psi_1\psi_1)\ldots(\zeta\bar\psi_n\psi_n) = (\zeta\bar\psi_1)\ldots(\zeta\bar\psi_n)\psi_n\ldots\psi_1 = \langle-\zeta\psi|n\rangle\langle n|\psi\rangle\,.\end{aligned}$$

2. In the coherent state representation, the quantum partition function of the oscillator Hamiltonian is expressed in terms of the path integral ($\hbar = 1$)

$$\mathcal{Z} = \int D(\bar{\phi}, \phi) \, \exp\left[- \int_0^\beta d\tau \left(\bar{\phi} \partial_\tau \phi + \omega \bar{\phi} \phi \right) \right] , \tag{4.42}$$

where $\phi(\tau)$ denotes a complex scalar field, the constant factor $e^{-\beta\omega/2}$ has been absorbed into the measure of the functional integral $D(\bar{\phi}, \phi)$, and we have set the chemical potential $\mu = 0$. The connection between the coherent state and Feynman integral is established by the change of field variables, $\phi(\tau) = \left(\frac{m\omega}{2}\right)^{1/2} \left(q(\tau) + \frac{ip(\tau)}{m\omega} \right)$, where $p(\tau)$ and $q(\tau)$ represent real fields. Substituting this representation in Eq. (4.42), and rearranging some terms by integrating by parts, the connection is established: $\mathcal{Z} = \int D(p,q) \, \exp[- \int_0^\beta d\tau (-ip\dot{q} + \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2)]$.

3. Making use of the Gaussian functional integral for complex fields, one obtains from Eq. (4.42) ($\hbar = 1$)

$$\begin{aligned} \mathcal{Z}_{\rm B} &\sim \det(\partial_\tau + \omega)^{-1} \sim \prod_{\omega_n} (i\omega_n + \omega)^{-1} \sim \prod_{n=1}^{\infty} \left[\left(\frac{2n\pi}{\beta} \right)^2 + \omega^2 \right]^{-1} \\ &\sim \prod_{n=1}^{\infty} \left[1 + \left(\frac{\beta\omega}{2\pi n} \right)^2 \right]^{-1} \sim \frac{1}{\sinh(\beta\omega/2)} \, . \end{aligned}$$

Now, in the limit of small temperatures, the partition function is dominated by the ground state, $\lim_{\beta\to\infty} \mathcal{Z}_{\rm B} = \exp[-\beta\omega/2]$, which fixes the constant of proportionality. Thus, $\mathcal{Z}_b = [2\sinh(\beta\hbar\omega/2)]^{-1}$.

In the fermionic case, the Gaussian integration obtains a product over eigenvalues in the numerator and we have to use fermionic Matsubara frequencies, $\omega_n = (2n+1)\pi/\beta$:

$$\begin{aligned} \mathcal{Z}_f &\sim \det(\partial_\tau + \omega) \sim \prod_{\omega_n} (i\omega_n + \omega) \sim \prod_{n=1}^{\infty} \left[\left(\frac{(2n+1)\pi}{\beta} \right)^2 + \omega^2 \right] \\ &\sim \prod_{n=1}^{\infty} \left[1 + \left(\frac{\beta\omega}{(2n+1)\pi} \right)^2 \right] \sim \cosh(\beta\omega/2) \, . \end{aligned}$$

Fixing the normalization, one obtains $\mathcal{Z}_f = 2e^{-\beta\omega}\cosh(\beta\omega/2)$. Altogether, these results are easily confirmed by direct computation, viz.

$$\begin{aligned} \mathcal{Z}_{\rm B} &= e^{-\beta\omega/2} \sum_{n=0}^{\infty} e^{-n\beta\omega} = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}} = \frac{1}{2\sinh(\beta\omega/2)} \, , \\ \mathcal{Z}_{\rm F} &= e^{-\beta\omega/2} \sum_{n=0}^{1} e^{-n\beta\omega} = e^{-\beta\omega/2}(1 + e^{-\beta\omega}) = 2e^{-\beta\omega}\cosh(\beta\omega/2) \, . \end{aligned}$$