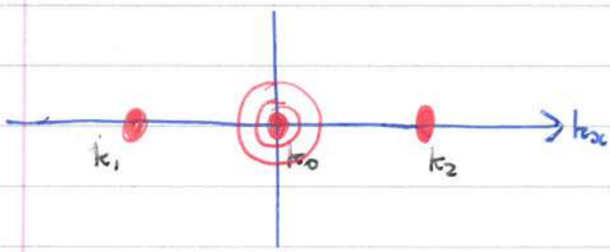
INTERACTIONS LECTURE

Now aim to construct Hamiltonian.
Contains two types of term:

$$H = \underbrace{KE}_{\text{single body}} + \underbrace{V_{e-i}}_{\text{"single body"}} + \underbrace{V_{e-e}}_{\text{two body}}$$

Single body:

Work in diagonal representation



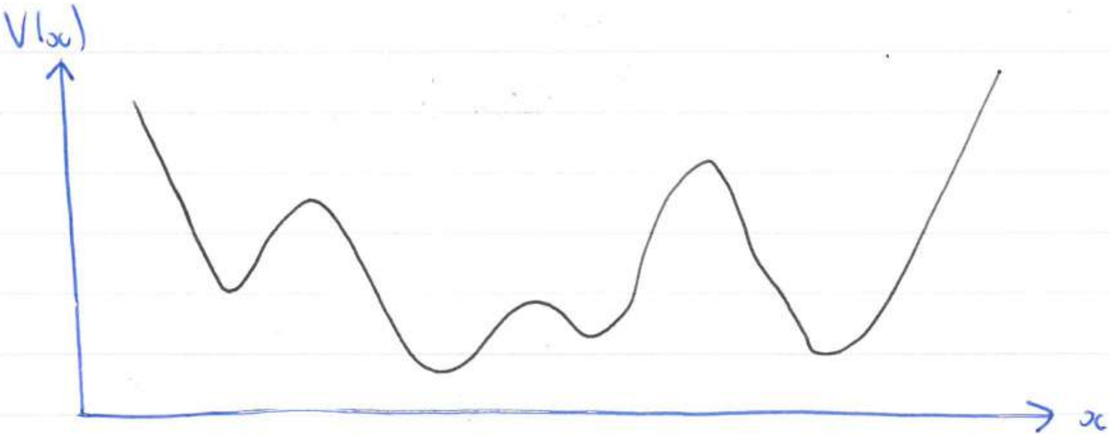
$$|\psi\rangle = (a^{+}_{k0})^3 \, a^{+}_{k1} \, a^{+}_{k2} \, |\Omega\rangle$$

Note: Fermions can only have one per state

$$\hat{KE} = \frac{\hbar^2}{2m} \sum_i k_i^2 . a^{+}_{ki} a_{ki}$$

$$KE|\psi\rangle = \frac{\hbar^2}{2m}\left(3k_0^2 + k_1^2 + k_2^2 + k_3^2 + k_4^2\right).|\psi\rangle$$

Count number of particles in state and multiply by eigenvalue of one-body operator



$$\hat{V} = \int_{-\infty}^{\infty} V(x)\, a^{+}(x)\, a(x)\, dx$$

If doing quantum harmonic oscillator KE + V now partly in momentum, part in position. To convert between the representations:

$$\hat{KE} = \sum_{p} \frac{\hbar^2 k^2}{2m} a_k^{+} a_k$$

But ladder operators linked through second quantization

$$a_k = \frac{1}{\sqrt{L}} \int_0^L e^{ikx} a(x)\, dx \qquad a_k^{+} = \frac{1}{\sqrt{L}} \int_0^L e^{-ikx} a^{+}(x)\, dx$$

so re-write in position representation

$$\hat{KE} = \frac{\hbar^2}{2mL}\sum_k k^2 \int_0^L\int_0^L e^{ik(x-x')}\, a^+(x')\, a(x)\, dx'\, dx$$

$$= \frac{\hbar^2}{2mL}\sum_k \int_0^L\int_0^L a^+(x')\, e^{-ikx'} \cdot -\frac{d^2}{dx^2}\left(e^{ikx}\right) a(x)\, dx'\, dx$$

$$= \frac{\hbar^2}{2mL}\sum_k \int_0^L\int_0^L a^+(x')\, e^{-ikx'}\, e^{ikx} \cdot -\frac{d^2}{dx^2}\left(a(x)\right) dx'\, dx$$

But $\hat{p} = -i\hbar\frac{d}{dx}$ so $\hat{p}^2 = -\hbar^2\frac{d^2}{dx^2}$ so

$$= \frac{1}{2mL}\sum_k \int_0^L\int_0^L a^+(x')\, e^{-ikx'}\, e^{ikx}\, \hat{p}^2\, a(x)\, dx'\, dx$$

$$= \int_0^L\int_0^L a^+(x')\, \frac{1}{L}\sum_k e^{ik(x-x')}\, \frac{\hat{p}^2}{2m}\, a(x)\, dx'\, dx$$

But $\frac{1}{L}\sum_k e^{ik(x-x')} = \delta(x-x')$ so

$$= \int_0^L\int_0^L a^+(x')\, \delta(x-x')\, \frac{\hat{p}^2}{2m}\, a(x)\, dx'\, dx$$

$$= \int_0^L a^+(x)\, \frac{\hat{p}^2}{2m}\, a(x)\, dx$$

Therefore the full Hamiltonian is

$$\hat{H} = \int_{-\infty}^{\infty} a^{+}(x) \left(\frac{\hat{p}^2}{2m} + V(x) \right) a(x)\, dx$$

Note: if in parabolic potential well:

$$\hat{H} = \int_{-\infty}^{\infty} a^{+}(x) \left(\frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \right) a(x)\, dx$$

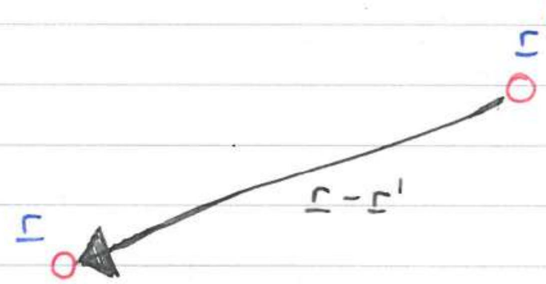
this is better written in the natural basis set of the harmonic oscillator

$$\hat{H} = \sum_n \hbar\omega \left(n + \frac{1}{2} \right) a_n^{+} a_n$$

Two body operators

Coulomb interaction

$$V(r) = \frac{e^2}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}$$



note: double counting

$$\hat{V} = \frac{1}{2}\int d\mathbf{r} \int d\mathbf{r}' \; c^+(\mathbf{r})\, c^+(\mathbf{r}')\, V(\mathbf{r}-\mathbf{r}')\, c(\mathbf{r}')\, c(\mathbf{r})$$

Examine integrand (for fermions)

$$V(\mathbf{r}-\mathbf{r}')\, c^+(\mathbf{r})\, c^+(\mathbf{r}')\, c(\mathbf{r}')\, c(\mathbf{r})$$

$$= -V(\mathbf{r}-\mathbf{r}')\, c^+(\mathbf{r})\, c^+(\mathbf{r}')\, c(\mathbf{r})\, c(\mathbf{r}') \qquad c^+c + cc^+ = \delta$$

$$= V(\mathbf{r}-\mathbf{r}')\, c^+(\mathbf{r}) \left(c(\mathbf{r})\, c^+(\mathbf{r}') - \delta(\mathbf{r}-\mathbf{r}') \right) c(\mathbf{r}') \qquad n = c^+c$$

$$= V(\mathbf{r}-\mathbf{r}')\, \hat{n}(\mathbf{r})\, \hat{n}(\mathbf{r}') - \delta(\mathbf{r}-\mathbf{r}')\, \hat{n}(\mathbf{r}) \neq V(\mathbf{r}-\mathbf{r}')\, \hat{n}(\mathbf{r})\, \hat{n}(\mathbf{r}')$$

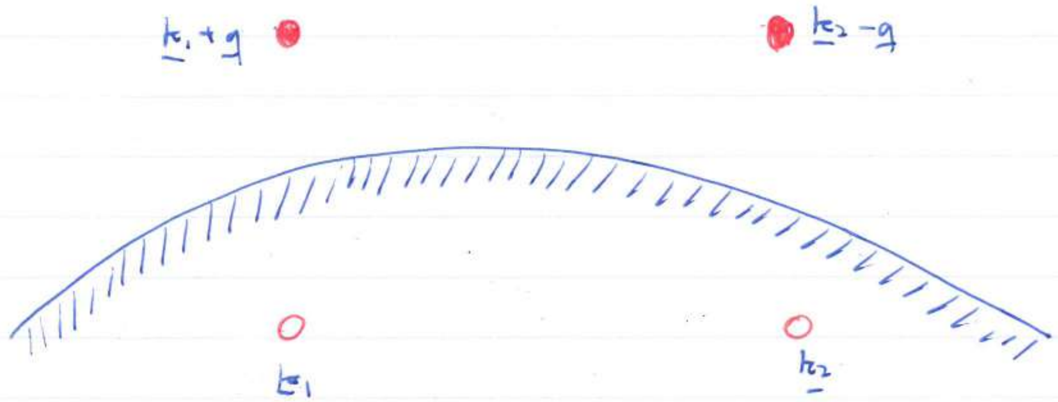
Differ from naive due to quantum mechanics
This term generates lasers

Fourier transform to same basis as kinetic energy / used to describe homogenous system:

$$\frac{1}{2}\int d\underline{r}\int d\underline{r}'\, c^+(\underline{r})c^+(\underline{r}')\, V(\underline{r}-\underline{r}')\, c(\underline{r}')\, c(\underline{r})$$

$$= \sum_{k_1,k_2,q} V(q)\, c^+_{\underline{k}_1+\underline{q}}\, c^+_{\underline{k}_2-\underline{q}}\, c_{\underline{k}_2}\, c_{\underline{k}_1}$$

Note: removal of momentum sum due to no com motion



$$|\Psi\rangle = \sum_{\underline{k}<\underline{k}_F} c^+_{\underline{k}}\,|\Omega\rangle$$