Fractional Chern Insulators in Harper-Hofstadter Bands with Higher Chern Number

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The Harper-Hofstadter model provides a fractal spectrum containing topological bands of any integer Chern number C. We study the many-body physics that is realized by interacting particles occupying Harper-Hofstadter bands with $|C| > 1$. We formulate the predictions of Chern-Simons or composite fermion theory in terms of the filling factor ν, defined as the ratio of particle density to the number of single-particle states per unit area. We show that this theory predicts a series of fractional quantum Hall states with filling factors $\nu = r/(r|C| + 1)$ for bosons, or $\nu = r/(2r|C| + 1)$ for fermions. This series includes a bosonic integer quantum Hall state in $|C| = 2$ bands. We construct specific cases where a single band of the Harper-Hofstadter model is occupied. For these cases, we provide numerical evidence that several states in this series are realized as incompressible quantum liquids for bosons with contact interactions.

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Recently, there has been much progress towards experimental realizations of topological flat bands, such as by light-matter coupling in cold gases [1–7] or via spin-orbit coupling in condensed matter systems [8–10]. These systems provide novel avenues for exploring fractional quantum Hall physics in new settings where lattice effects play important roles [8–19]. Furthermore, these "fractional Chern insulators" generalize the fractional quantum Hall states of interacting particles in continuum Landau levels to lattice-based systems.

In cases where the underlying topological band has unit Chern number $C = 1$ the states can be continuously connected to conventional fractional quantum Hall states in the continuum Landau level [20]. However, if the band has Chern number C of magnitude greater than 1, no such continuity is possible. The fractional quantum Hall states have features that are particular to the lattice structure. The appearance of fractional quantum Hall states for bands with $|C| > 1$ has been demonstrated in various lattice models, with unit cells that contain multiple states in the form of distinct sublattices, or in terms of internal degrees of freedom such as spin or color (see, e.g., Ref. [21]). In particular, this has led to the proposal of states at filling factors $\nu = 1/(|C| + 1)$ for bosons [22–25].

In this Letter we show that this physics of interacting particles in the novel Chern bands can be captured within the Harper-Hofstadter model, in which a magnetic unit cell arises naturally without additional internal degrees of freedom. This model leads to a complex energy spectrum as a function of flux $n_\phi = \Phi/\Phi_0$ per plaquette. The low energy bands can have Chern numbers larger than 1. We show that they can realize the sequences of fractional Chern insulator states for $|C| > 1$ discussed for other models, providing an interpretation of these states in terms of the composite fermion construction on a lattice [11,14]. Based on these insights, we identify another sequence of fractional Chern insulator states with filling factors $\nu = r/(r|C| + 1)$. We show numerical evidence for this sequence from exact diagonalization studies.

The Harper-Hofstadter model has recently been realized, at least for weakly interacting particles, in experiments on ultracold gases [2–4]. Further realizations have also been obtained for a triangular moiré lattice in graphene flakes deposited on boron nitride [26–28]. Our results demonstrate that, under suitable conditions (particle density, flux density, and temperature), these systems have the possibility to explore a wide range of the novel physics of fractional Chern insulators.

The rich physics of charged particles in a two-dimensional plane subjected to a perpendicular magnetic field B and a periodic potential, or physically equivalent models emulating this scenario, arises from its two competing length scales: the magnetic length $\ell_0 = \sqrt{\hbar/eB}$ and the lattice scale a. Hence, the problem brings to play the commensurability of these two scales, as first analyzed by Harper [29]. The resulting fractal structure of the single-particle spectrum was revealed by Azbel [30] and the full spectrum solved numerically and characterized by Hofstadter [31]. As shown by Wannier [32], the Azbel-Hofstadter recursion relations imply that the total number of states per unit area n_s below each gap varies linearly with the flux density n_ϕ, and further that their relationship is described by the Diophantine equation

$$n_s = Cn_\phi - D, \qquad C, D \in \mathbb{Z}. \tag{1}$$

The work by Streda [33], as well as Thouless *et al.* [34,35], explains the physical relevance of Wannier's result. For noninteracting fermions, one expects incompressible states at density $n = n_s$, with Hall conductivity given by Streda's formula as

$$\sigma_{xy} = \frac{e}{\Phi_0}\frac{\partial n}{\partial n_\phi} = C\frac{e^2}{h}. \quad (2)$$

Thouless *et al.* derive the Hall conductivity by direct calculation from a Kubo formula, obtaining the integer quantization from the topological nature of the resulting expression, namely, that $C = \sum \mathcal{C}_i$, where $\mathcal{C}_i$ is the Chern number of the ith occupied band [34]. Thus, the integer C in Eq. (1) is seen to be the net Chern number of the bands contributing to the n_s states below the energy gap. Solutions to Eq. (1) exist for any C in which n_s arises from a single band (see below), establishing the presence of bands of any Chern number in the Hofstadter spectrum (albeit with rapidly decreasing gaps for large $\mathcal{C}$) [36,37].

To realize the nontrivial Chern bands of the Harper-Hofstadter model, it is sufficient to create a tight-binding lattice system with complex hopping elements between nearest neighbor sites, a fact exploited for realizations of the model in cold gases [2,3,6]. From the analysis of the magnetic translation group [38,39], it follows that at a rational flux density $n_\phi = p/q$, the Harper-Hofstadter Hamiltonian admits a periodic representation on a magnetic unit cell (MUC) comprising q sites, i.e., an area enclosing an integer number of flux quanta and an integer number of plaquettes of the lattice. The single-particle Hamiltonian then takes the tight-binding form

$$\mathcal{H}_{\rm sp} = -\sum_{i,j} t_{ij} e^{\phi_{ij}} \hat{a}_j^\dagger \hat{a}_i + \text{H.c.}, \quad (3)$$

in which the phases ϕ_{ij} are invariant under translations of the MUC. (The choice of the MUC and the vector potential $\mathbf{A}$ make up the remaining space of gauge choices.) One can therefore consider the q sites of the MUC as sublattices $\alpha = 1, \ldots, q$ of a general q-site tight-binding model and solve via Bloch's theorem. Note that the origin of the MUC can be chosen on any site of the lattice, so the problem has an additional q-fold symmetry. We give an analysis of the single-particle properties in the Supplemental Material [40], taking care to respect this symmetry [41].

To search for strongly correlated phases, it is useful to identify situations in which there is a manifold of low-energy single-particle states (one band, or several closely spaced bands) that is well separated from higher-energy bands. For now, we focus on the case in which this manifold is a single band, with a large gap to the next band. The largest gap in the Harper-Hofstadter spectrum corresponds to the lowest Landau level, with $C = 1$, $D = 0$. Here, we seek more general states, and consider the next-largest gaps found at the first level of the Hofstadter hierarchy. These appear in the vicinity of cell boundaries close to the simple rational flux densities $n_\phi = 1/Q$, $(Q > 1)$, where the energy bands become exponentially flat in terms of both their energy dispersion and their Berry curvature. Hence, these bands are well suited to support incompressible fractional quantum Hall states [13,14,53,54]. The gaps near points $(n_s, n_\phi) = (0, Q^{-1})$ are described by the Diophantine equation (1) with $C \equiv sQ$, and $D = \text{sgn}(C) = s$, with $s = \pm 1$ for the bands at $n_\phi \gtrless 1/Q$. We are particularly interested in cases where we find a single band below this gap, i.e., where $n_s(n_\phi = p/q) = 1/q$, and thus $q = Qp - s$, with corresponding flux densities

$$n_\phi = \frac{p}{|C|p - \text{sgn}(C)}, \qquad p \in \mathbb{N}. \quad (4)$$

Below, we take $p > 2$ to ensure that the band belongs to the subcell nearest $n_\phi = 1/|C|$. We will also consider higher band gaps at the same flux densities, which can be seen as fractal replicas of the rth continuum Landau level, for which

$$n_s = r(Cn_\phi - \text{sgn}C), \qquad r \in \mathbb{Z}\backslash\{0\}, \quad (5)$$

and where $rC \gtrless 0$ for the flux densities $n_\phi \gtrless 1/Q$.

We now discuss the many-body physics of interacting particles in the Harper-Hofstadter model, described by the Hamiltonian

$$\mathcal{H} = \mathcal{H}_{\rm sp} + \frac{1}{2}\sum_{i,j} V(\mathbf{r}_i - \mathbf{r}_j) :\hat{n}_i \hat{n}_j:, \quad (6)$$

with site labels i, j, and $:\hat{n}:$ denoting normal ordering of the density operators. Let us first review the predictions of Chern-Simons theory [11], or, equivalently, the lattice composite fermion picture [14], and translate these results into the language used for the analysis of Chern insulators. The basic premise of this approach is that the interaction includes a sufficiently strong short-range repulsion in order to favor "flux attachment," which keeps the particles at a distance from each other, thus minimizing interaction energy. The composite fermion ansatz translates this idea into a trial wave function of the form $\Psi_{\rm trial}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \mathcal{P}_{\rm LEM}\Psi_J(\{\mathbf{r}_i\}) \times \Psi_{\rm CF}(\{\mathbf{r}_i\})$, where both the Jastrow factor Ψ_J and the composite fermion wave function $\Psi_{\rm CF}$ vanish when the positions of two particles coincide, and $\mathcal{P}_{\rm LEM}$ denotes the projection onto the relevant low-energy manifold of single-particle states. For the case of bosons (fermions), one needs to attach an odd (even) number k of flux quanta to the particles, so as to obtain an effective problem of weakly interacting composite fermions (CF) experiencing an effective flux density n_ϕ^* relating to the externally applied flux via

$$n_\phi = kn + n_\phi^*, \qquad k \in \mathbb{Z}\backslash\{0\}. \quad (7)$$

If the CFs behave as weakly interacting particles, they will form incompressible (topological) insulating states when filling an integer number of bands. Their band structure is given by a Harper-Hofstadter Hamiltonian with flux density n_ϕ^*. The densities n_s^* at which filled bands are realized are therefore given by a Diophantine equation of the form (1) for the composite fermion system, $n_s^* = C^* n_\phi^* - D^*$, with integer parameters C^*, D^*. In composite fermion theory, one can explain the fractional QHE as an integer QHE of composite fermions [55], and for the lattice case one thus predicts incompressible states at $n = n_s^*$. Hence, with Eq. (7), we find

$$n = \frac{C^* n_\phi - D^*}{kC^* + 1} > 0. \tag{8}$$

Choices of the parameters C^* and D^* for the composite fermion gap yield various candidates for incompressible states in the spectrum, given in terms of density, as illustrated previously as Fig. 1 in Ref. [14].

In order to relate the densities (8) to the "filling factor" of FQH systems there are several choices that can be made. One choice is to consider the ratio $\nu_\phi \equiv n/n_\phi$ of particle density to flux density, which is natural in the continuum limit $n_\phi \to 0$, where the bands of the Harper-Hofstadter model reduce to continuum Landau levels. More generally, and to allow connections to fractional Chern insulator models, the natural filling factor to consider is the ratio $\nu \equiv n/n_s$ of the particle density to the number density of single-particle states in the low-energy manifold (e.g., the lowest energy band if this manifold is a single band). We replace n_ϕ in Eq. (8) via Eq. (1) and obtain

$$n\left(\frac{1 + kC^*}{C^*}\right) + \frac{D^*}{C^*} = \frac{n_s}{C} + \frac{D}{C}. \tag{9}$$

In general, all parameters $\{k, C, C^*, D, D^*\}$ contribute to determine eligible states. However, as we now describe, for some important cases the ratios $D^*/C^* = D/C$ are equal, and the states can be characterized by a fixed filling factor $\nu = n/n_s$.

It is instructive to consider special cases. First, the fractional quantum Hall states in a $C = 1$ band are recovered by choosing the manifold as the (lattice equivalent of the) lowest Landau-level ($C = 1$, $D = 0$), and taking general integers $C^* = t$, $D^* = 0$ for filling composite fermion states in the tth Landau level. One recovers the usual Jain series of states $\nu = t/(kt + 1)$. (In this case, $n_s = n_\phi$, so the two definitions of filling factor coincide.)

Second, we can take both the CF bands and the effective low-energy bands in the same subcell of the Hofstadter spectrum, close to the flux $n_\phi = 1/|C|$. As a concrete example, consider the gaps (5) and choose to fill r bands of composite fermions such that $C^* = rC$, and $D^* = r\mathrm{sgn}(C)$, and choose the lowest band with the given C, and $D = \mathrm{sgn}(C)$ for the manifold of single-particle states. We obtain states with filling factors [56]

$$\nu^{C^*=rC} = \frac{n}{n_s} = \frac{r}{r|kC| + 1}, \qquad r \in \mathbb{Z}\backslash\{0\}, \tag{10}$$

where states with $r < 0$ represent the generalization of negative flux attachment. In general, the low-energy manifold supporting these states will have many bands, but in the cases (4) this reduces to a single band.

The sequence of filling factors (10), valid for any Chern number $C \neq 0$, is a core result of our Letter [57].

Several remarks are in order. The case with $r = 1$ can be seen as an analogue of the Laughlin state, in the sense that a single band of composite fermion states is filled. From the previous studies of the Laughlin state on the lattice in a $C = 1$ band, we can infer useful intuition on the likely stability of such states. The Laughlin state was shown to be stable up to flux densities $n_\phi \simeq 0.4$; i.e., it persists through 80% of the region in which a gap is open [60]. Likewise, the $\nu = 2/3$ hierarchy state was seen to be stable up to $n_\phi \simeq 0.3$ [14]. In the case of the states stabilized in subcells with $|C| > 1$ bands, we note that the bands tend to have less dispersion, albeit maybe larger fluctuations in the band geometry. By analogy, the family of states (10) can be expected to be stable at a substantial distance from the respective cell boundaries.

The reader will note that the prediction of composite fermion theory (10) includes the Abelian states at filling factors $\nu = 1/(|kC| + 1)$ that have recently been described in studies of Chern bands with Chern number $|C| > 1$ [22–24], for both bosons and fermions, and which were described in terms of C flavor states [13,25,53,54]. The derivation presented here demonstrates that these states are predicted also by the concept of flux attachment [11] leading to CF wave functions of the form described in Ref. [14]. While the C-flavor or multilayer language appears to require C copies of a $C = 1$ Brillouin zone, implying finite size geometries with a number of states $N_s \mathrm{mod} C = 0$, it was shown that a color-entangled formulation remedies this constraint [25]. Note that the hierarchy wave functions following from the CF construction [14] similarly do not require any constraint on the lattice geometry.

The composite fermion theory makes a more general prediction, in that it does not require that the single particle states making up the manifold n_s are from a single band, as n_ϕ can vary continuously in Eq. (9). Indeed, perturbing a stable quantum liquid formed in a single-band configuration by an infinitesimal change in n_ϕ, the low-energy manifold splits up into (possibly infinitely) many bands, but we expect that the physics of the phase should be robust under this perturbation, providing a notion of adiabatic continuity that allows us to connect any band to the limit of the perfectly flat general Chern bands obtained as $n_\phi \to 1/C$ [40], in line with the behavior seen in $C = 1$ Harper-Hofstadter bands [61].

The filling factors (10) are analogous to the hierarchy states, which have been observed in Chern number one fractional Chern insulators [14,42,62]. Their properties, in terms of quasiparticle charges and statistics were predicted by Kol and Read [11], as summarized in Ref. [40]. Unlike the lowest Landau level, $|C| > 1$ bands support states with negative flux attachment ($r < 0$) even for $|k| = 1$, $r = -1$, so the corresponding series of states leads to novel filling factors. Numerical evidence for the $\nu = 1$ state in a $C = 2$ band obtained for $Ckr = -2$ was provided by the current authors in Ref. [14], which is a special case in that it realizes an integer quantum Hall effect of bosons [63].

The limit of filling many CF Landau levels, $r \to \infty$, which represents the equivalent of the half-filled Landau level, converges to

$$\lim_{r\to\infty} \nu^{C^*=rC} = \frac{1}{|kC|}. \tag{11}$$

At these points, the composite fermion spectrum resembles a Fermi sea, as the band gaps between the composite fermion levels decrease as $1/r$ and evolve into a quasi-continuum. In analogy to the half-filled continuum Landau levels, one may ask whether this filling can be susceptible to the equivalent of a CF pairing instability, or possibly more exotic states. In the $C = 1$ case, the possibility of a Moore-Read state at $\nu = 1$ is well known [43]. For the $C = 2$ band near $n_\phi = 1/2$, a paired phase has been described in a related continuum model [54,65], though the model does not provide a quantitative description of $C = 2$ bands [44]. If realized, it is expected that a paired phase at the fillings (11) would be non-Abelian in Chern bands with $|C|$ *odd*, while Majorana quasiparticles will likely pair up for even Chern bands and thus recombine to yield an Abelian phase [65].

A case that has not yet been explored is the Abelian series of states (10). We examine the evidence for the presence of these composite fermion or hierarchy states on the basis of the band-projected Hamiltonian within the low-energy manifold, focusing on the single-band cases (4). The corresponding Hamiltonian, $\mathcal{H}_{\text{proj}} = \mathcal{P}_{\text{LEM}}\mathcal{H}\mathcal{P}_{\text{LEM}}$, can be studied in the same framework as other fractional Chern insulator models [17]. The residual dispersion of bands in the low-energy manifold could be of interest for studying phase transitions between fractional quantum Hall liquids and condensed phases of bosons or Fermi-liquidlike states of fermions, respectively. However, here we choose to neglect the residual band dispersion, particularly as it vanishes quickly as $n_\phi \to 1/|C|$ [40]. Furthermore, we focus on the case of bosons with contact interactions $V_{ij} = U\delta_{ij}$.

Our numerical study shows evidence supporting the existence of gapped quantum liquids at several filling factors of the series (10). First, states are found for the cases $r = 1$, $|k| = 1$, where the predictions of the filling factor $\nu = 1/(|C| + 1)$ coincide between the composite

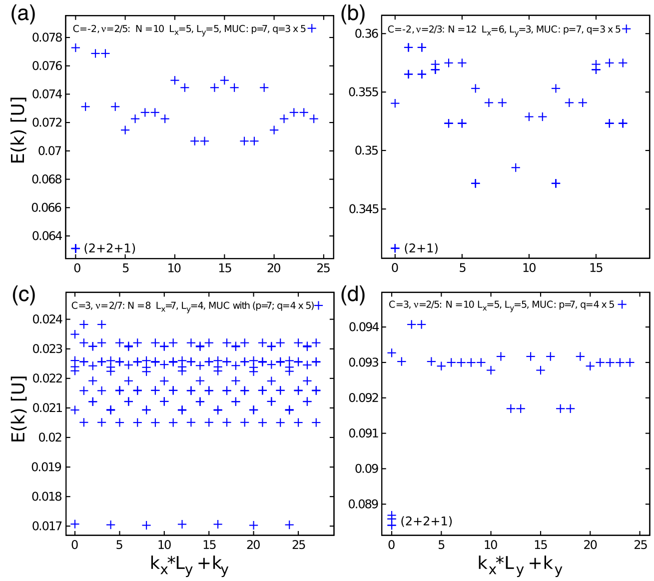

FIG. 1 (color online). Spectra for Chern insulators of the series (10) in Harper-Hofstadter bands at flux densities from Eq. (4). For Chern number $|C| = 2$ we choose flux density $n_\phi = 7/15$, showing (a) $\nu = 2/5$ for $N = 10$ (b) $\nu = 2/3$ for $N = 12$ bosons. For $C = 3$ we take $n_\phi = 7/20$, and show (c) $\nu = 2/7$ for $N = 8$, (d) $\nu = 2/5$ for $N = 10$. For lattice geometries, see legend. Multiplet structure of ground states shown in parentheses.

fermion theory and the analyses in terms of Halperin multicomponent [13,53,54] or color-entangled states [25]. The integer bosonic quantum Hall state with $r = -1$ was discussed in Ref. [14]. Here, we present evidence for additional states, such as those with $|r| = 2$, with two filled composite fermion bands. In Fig. 1, we show their spectra for Chern bands with $|C| = 2$ [panels (a) and (b)], and $C = |3|$ [panels (c) and (d)]. All cases show the correct ground state degeneracies predicted by CF theory ($d_{\text{GS}} = |1 + kC^*|$, see Refs. [11,40]). The states with positive flux attachment, $\nu = 2/5$ ($|C| = 2$) and $\nu = 2/7$ ($|C| = 3$) have the clearest signature in terms of the magnitude of the gap to the average state spacing of excitations. The states with negative flux attachment, $\nu = 2/3$ ($|C| = 2$) and $\nu = 2/5$ ($|C| = 3$) also show a distinct separation of energy scales.

A finite size scaling of the gap gives us further indications of the stability of these phases. Figure 2 shows the gap scaling for several filling factors in the same $|C| = 2$ and $|C| = 3$ bands as above. All systems we examined show the expected ground state degeneracy. In both bands, the largest gap is found for the $r = 1$ state of the series (10), while the $r = \pm 2$ states have a slightly smaller gap in the finite size systems. The extrapolation to the thermodynamic limit is consistent with a nonzero gap for both the $r = 1$ and $r = \pm 2$ states for $|C| = 2$. Data for the $C = 3$ cases are both noisier and include fewer system sizes. Nonetheless, the results are consistent with a nonzero gap.

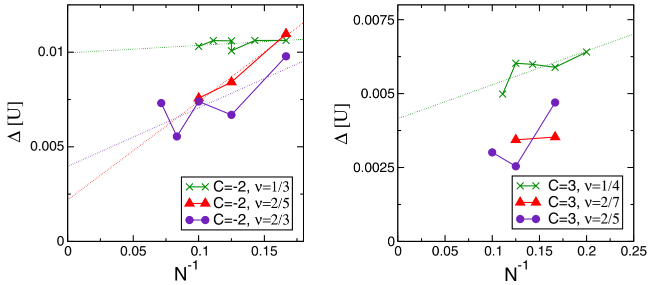

FIG. 2 (color online). Finite size scaling of the gap for states of the series (10), with $r = +1, +2, -2$. Left: a $C = -2$ band ($n_\phi = 7/15$) for states at $\nu = 1/3$, $\nu = 2/5$, and $\nu = 2/3$. Right: Finite size scaling for $C = 3$ states ($n_\phi = 7/20$) at $\nu = 1/4$, $\nu = 2/7$, and $\nu = 2/5$.

In the Supplemental Material, we briefly discuss particle entanglement spectra of our states, as well as spectral flow under flux insertion [40]. In addition to these ground state properties, we have examined spectra under addition of "flux," i.e., under changes of system size at fixed N. We find low-lying bands of states consistent with an interpretation as quasiparticle states of an underlying quantum Hall liquid. We leave the detailed analysis of these features for a future publication.

In conclusion, we have translated the composite-fermion or Chern-Simons theory into the language of fractional Chern insulators, leading to the prediction of a series of states with filling factors $\nu = r/(r|kC| + 1)$, for bosons ($|k| = 1$) or fermions ($|k| = 2$). This includes, and provides an alternative description for, the series of states $\nu = 1/(|kC| + 1)$ that were observed in the literature on FCI for $|C| > 1$. We have identified flux densities where a single isolated band of Chern number C occurs at the bottom of the Hofstadter spectrum. Finally, we have studied the many-body states of bosons with contact interactions under the projection into these Chern bands, identifying gapped states with the ground state degeneracies predicted by theory. While previous evidence had been given for the bosonic integer Chern insulator state with $\nu = 1$, $r = -1$, $|C| = 2$ [14], which was obtained for a hard-core interaction and without applying a band projection, the current results provide evidence for the wider applicability of composite fermion theory, and its validity also for the band-projected Hamiltonian. Our results can be extended to general $|C| > 1$ bands in other tight-binding models, and to the effective continuum limit $n_\phi \to 1/|C|$ via principles of adiabatic continuity. Further investigations should focus on the stability of fermionic states, the role of long-range interactions and the detailed analysis of the ground states and excitations in terms of the composite fermion trial wave functions.

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[1] Y. Lin, R. Compton, K. Jimenez-Garcia, J. Porto, and I. B. Spielman, Nature (London) **462**, 628 (2009).
[2] M. Aidelsburger, M. Atala, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, Phys. Rev. Lett. **111**, 185301 (2013).
[3] H. Miyake, G. A. Siviloglou, C. J. Kennedy, W. C. Burton, and W. Ketterle, Phys. Rev. Lett. **111**, 185302 (2013).
[4] M. Aidelsburger, M. Lohse, C. Schweizer, M. Atala, J. T. Barreiro, S. Nascimbène, N. R. Cooper, I. Bloch, and N. Goldman, Nat. Phys. **11**, 162 (2015).
[5] G. Jotzu, M. Messer, R. Desbuquois, M. Lebrat, T. Uehlinger, D. Greif, and T. Esslinger, Nature (London) **515**, 237 (2014).
[6] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, Rev. Mod. Phys. **83**, 1523 (2011).
[7] N. Goldman, G. Juzeliunas, P. Öhberg, and I. B. Spielman, Rep. Prog. Phys. **77**, 126401 (2014).
[8] E. Tang, J.-W. Mei, and X.-G. Wen, Phys. Rev. Lett. **106**, 236802 (2011).
[9] T. Neupert, L. Santos, C. Chamon, and C. Mudry, Phys. Rev. Lett. **106**, 236804 (2011).
[10] K. Sun, Z.-C. Gu, H. Katsura, and S. Das Sarma, Phys. Rev. Lett. **106**, 236803 (2011).
[11] A. Kol and N. Read, Phys. Rev. B **48**, 8890 (1993).
[12] A. S. Sørensen, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **94**, 086803 (2005).
[13] R. N. Palmer and D. Jaksch, Phys. Rev. Lett. **96**, 180407 (2006).
[14] G. Möller and N. R. Cooper, Phys. Rev. Lett. **103**, 105303 (2009).
[15] E. Kapit and E. Mueller, Phys. Rev. Lett. **105**, 215303 (2010).
[16] D. N. Sheng, Z.-C. Gu, K. Sun, and L. Sheng, Nat. Commun. **2**, 389 (2011).
[17] N. Regnault and B. A. Bernevig, Phys. Rev. X **1**, 021014 (2011).
[18] E. J. Bergholtz and Z. Liu, Int. J. Mod. Phys. B **27**, 1330017 (2013).
[19] S. A. Parameswaran, R. Roy, and S. L. Sondhi, C.R. Phys. **14**, 816 (2013).
[20] T. Scaffidi and G. Möller, Phys. Rev. Lett. **109**, 246805 (2012).
[21] Y.-H. Wu, J. K. Jain, and K. Sun, Phys. Rev. B **91**, 041119 (2015).
[22] Y.-F. Wang, H. Yao, C.-D. Gong, and D. N. Sheng, Phys. Rev. B **86**, 201101 (2012).
[23] Z. Liu, E. J. Bergholtz, H. Fan, and A. M. Läuchli, Phys. Rev. Lett. **109**, 186805 (2012).
[24] A. Sterdyniak, C. Repellin, B. A. Bernevig, and N. Regnault, Phys. Rev. B **87**, 205137 (2013).

[25] Y.-L. Wu, N. Regnault, and B. A. Bernevig, Phys. Rev. Lett. **110**, 106802 (2013).

[26] C. R. Dean, L. Wang, P. Maher, C. Forsythe, F. Ghahari, Y. Gao, J. Katoch, M. Ishigami, P. Moon, M. Koshino, T. Taniguchi, K. Watanabe, K. L. Shepard, J. Hone, and P. Kim, Nature (London) **497**, 598 (2013).

[27] L. A. Ponomarenko, R. V. Gorbachev, G. L. Yu, D. C. Elias, R. Jalil, A. A. Patel, A. Mishchenko, A. S. Mayorov, C. R. Woods, J. R. Wallbank, M. Mucha-Kruczynski, B. A. Piot, M. Potemski, I. V. Grigorieva, K. S. Novoselov, F. Guinea, V. I. Fal'ko, and A. K. Geim, Nature (London) **497**, 594 (2013).

[28] B. Hunt, J. D. Sanchez-Yamagishi, A. F. Young, M. Yankowitz, B. J. LeRoy, K. Watanabe, T. Taniguchi, P. Moon, M. Koshino, P. Jarillo-Herrero, and R. C. Ashoori, Science **340**, 1427 (2013).

[29] P. G. Harper, Proc. Phys. Soc. London Sect. A **68**, 874 (1955).

[30] M. Y. Azbel, ZhETF **46**, 929 (1964) [J. Exp. Theor. Phys. **19**, 634 (1964)].

[31] D. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

[32] G. H. Wannier, Phys. Status Solidi B **88**, 757 (1978).

[33] P. Streda, J. Phys. C **15**, L717 (1982).

[34] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).

[35] Q. Niu, D. J. Thouless, and Y.-S. Wu, Phys. Rev. B **31**, 3372 (1985).

[36] D. Osadchy and J. E. Avron, J. Math. Phys. (N.Y.) **42**, 5665 (2001).

[37] J. E. Avron, D. Osadchy, and R. Seiler, Phys. Today **56**, 38 (2003).

[38] P. G. Harper, Proc. Phys. Soc. London Sect. A **68**, 879 (1955).

[39] J. Zak, Phys. Rev. **136**, A776 (1964).

[40] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.115.126401 for the construction of the single-particle Harper-Hofstadter Hamiltonian, a discussion of the Berry curvature in the quasi-continuum limits $n_\phi \to 1/|C|$, the role of adiabatic continuity within Hofstadter gaps and in relation to other models, a review of properties of composite fermion states, data on particle entanglement spectra and spectral flow, and additional Refs. [11,14,20,24,25,41,42,43,44–52].

[41] T. S. Jackson, G. Möller, and R. Roy, arXiv:1408.0843v1.

[42] T. Liu, C. Repellin, B. A. Bernevig, and N. Regnault, Phys. Rev. B **87**, 205136 (2013).

[43] A. Sterdyniak, N. Regnault, and G. Möller, Phys. Rev. B **86**, 165314 (2012).

[44] F. Harper, S. H. Simon, and R. Roy, Phys. Rev. B **90**, 075104 (2014).

[45] J. Zak, Phys. Rev. **134**, A1602 (1964).

[46] Y.-H. Wu, J. K. Jain, and K. Sun, Phys. Rev. B **86**, 165129 (2012).

[47] T. Scaffidi and S. H. Simon, Phys. Rev. B **90**, 115132 (2014).

[48] S. A. Parameswaran, R. Roy, and S. L. Sondhi, Phys. Rev. B **85**, 241308 (2012).

[49] R. Roy, Phys. Rev. B **90**, 165139 (2014).

[50] M. Barkeshli and X.-L. Qi, Phys. Rev. X **2**, 031013 (2012).

[51] Y.-L. Wu, N. Regnault, and B. A. Bernevig, Phys. Rev. B **86**, 085129 (2012).

[52] Y.-L. Wu, N. Regnault, and B. A. Bernevig, Phys. Rev. B **89**, 155113 (2014).

[53] R. N. Palmer, A. Klein, and D. Jaksch, Phys. Rev. A **78**, 013609 (2008).

[54] L. Hormozi, G. Möller, and S. H. Simon, Phys. Rev. Lett. **108**, 256809 (2012).

[55] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).

[56] Note, the positivity of Eq. (8) implies $kC > 0$, which fixes the required sign of k.

[57] Note that the physics in time-reversal symmetric bands with Chern number $C = 0$ yields very different phenomena, such as Bose condensates [58] or supersolids [59].

[58] G. Möller and N. R. Cooper, Phys. Rev. A **82**, 063625 (2010).

[59] G. Möller and N. R. Cooper, Phys. Rev. Lett. **108**, 045306 (2012).

[60] M. Hafezi, A. S. Sørensen, E. Demler, and M. D. Lukin, Phys. Rev. A **76**, 023613 (2007).

[61] D. Bauer, T. S. Jackson, and R. Roy, arXiv:1504.07185v1 [Phys. Rev. Lett. (to be published)].

[62] A. M. Läuchli, Z. Liu, E. J. Bergholtz, and R. Moessner, Phys. Rev. Lett. **111**, 126802 (2013).

[63] We thank B. Béri for pointing out a possible relation to the boson integer quantum Hall effect (bIQHE) [64].

[64] T. Senthil and M. Levin, Phys. Rev. Lett. **110**, 046801 (2013).

[65] G. Möller, L. Hormozi, J. Slingerland, and S. H. Simon, Phys. Rev. B **90**, 235101 (2014).